SOLUTIONS - FIRST HOUR TEST - EEL 3473 - Sept 25, 2009

1. Discuss the concept of a "skin depth" in no more than one page.

In any material with constant conductivity σ and permittivity ϵ, the phasor (one frequency) plane wave electric field is

$$\hat{\vec{E}} = E_0^+ e^{-\alpha z} e^{-j\beta z} \quad \text{in the } +z \text{ direction}$$

We define the skin depth $\delta_s = \frac{1}{\alpha}$, so

$$\hat{E}(z) = E_0^+ e^{-z/\delta_s} e^{-j\beta z}$$

Thus, when the wave has propagated $z = \delta_s$ the amplitude is decreased $e^{-1} = \frac{1}{e} \simeq 1/3$

For a general material with σ, ϵ

$$\alpha = \frac{1}{\delta_s} = \omega \left(\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon}\right)^2} - 1 \right] \right)^{1/2}$$

with $\epsilon'' = \sigma/\omega$

For a good conductor, the most common situation in which a skin depth δ_s is used,

$$\frac{\epsilon''}{\epsilon} = \frac{\sigma}{\omega\epsilon} \gg 1$$

and hence $\alpha = \frac{1}{\delta_s} = \sqrt{\pi f \mu \sigma}$ where μ is the permeability

The skin depth for a good conductor is inversely proportional to the square root of frequency and of conductivity, $\delta_s = \frac{1}{\sqrt{\pi \mu f \sigma}}$

In a good conductor the $\vec{J} \cdot \vec{E}$ (I^2R) heating losses cause the Poynting vector $\vec{S}$ to decrease $\sim (e^{-z/\delta_s})^2$ with z

2. Discuss the relationship between value of the relaxation time, $\tau = \varepsilon / \sigma$, for charge dispersal in a material and the characteristics of the plane waves that can propagate in that material. Use less than one page.

In a material with constant σ and constant ϵ, any group of charges will disperse with time as $e^{-t/\tau}$ with $\tau = \epsilon/\sigma$. The dispersion (or decrease) is driven by the electric field of the charges themselves.

When a propagating electromagnetic wave traverses the material, its electric field will also drive charge motion. If the period of the wave T ($T = 1/f$) is small compared to τ, there is no time for any charge to interact with itself during many periods of the wave and the material appears to the wave to be an insulator $T = \frac{1}{f} \ll \tau = \epsilon/\sigma$ or $\boxed{\sigma \ll \omega\epsilon}$ where $\omega = 2\pi f$

For the wave

$$\nabla \times \tilde{\vec{H}} = \underbrace{\sigma \tilde{\vec{E}}}_{\text{cond. current}} + \underbrace{j\omega\epsilon \tilde{\vec{E}}}_{\text{displacement current}}$$

so, if $\sigma \ll \omega\epsilon$

$$\nabla \times \vec{H} \doteq j\omega\epsilon\vec{E}$$, and the wave phase vel $= \frac{1}{\sqrt{\mu\epsilon}}$

Conversely, if $T \gg \tau$, the charge in the material has time to react before the wave amplitude changes and $\sigma \gg \omega\epsilon$, $\nabla \times \tilde{H} = \sigma \tilde{E}$, and the Maxwell eqns sol. for $\tilde{E}$ is

$$\tilde{E} = E_0^+ e^{-z/\delta_s} e^{-jz/\delta_s}, \quad \delta_s = \frac{1}{\sqrt{\pi\mu f \sigma}}$$

$$\vec{E}(z,t) = E_0^+ e^{-z/\delta_s} \cos(\omega t - z/\delta_s) \quad \text{with } v \ll \frac{1}{\sqrt{\mu\epsilon}}$$

3. Define the Poynting vector and sketch (or discuss) how it is derived. Use less than one page.

In the time domain

$$\vec{S}(\bar{r},t) = \vec{E}\times\vec{H} \quad \text{watts}/m^2 \quad , \vec{E}(\bar{r},t), \vec{H}(\bar{r},t)$$

In the phasor domain

$$\tilde{\vec{S}}(\vec{r}) = \hat{\vec{E}}\times\tilde{\vec{H}}^* \quad , \quad \vec{S}(\vec{r},t) = \frac{1}{2}\,\mathrm{Re}\left(\hat{\vec{E}}\times\hat{\vec{H}}^*\right)$$

for one frequency

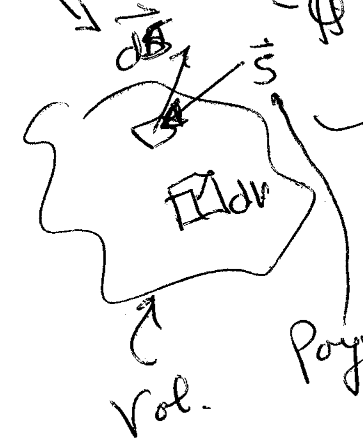
$\vec{S}$ is the energy in the wave passing per unit area in unit time

$-\oiint \vec{E}\times\vec{H}\cdot d\vec{s}$ is the total energy through the closed surface surrounding any volume

It can be shown from Maxwell's curl equations that

$$-\oiint \underbrace{\vec{E}\times\vec{H}\cdot d\vec{s}}_{\text{power into volum}} = \underbrace{\iiint \bar{E}\cdot\frac{d\bar{D}}{dt}\,dV}_{\text{increase with time of electric energy storage}} + \underbrace{\iiint \vec{H}\cdot\frac{\partial\bar{B}}{\partial t}\,dV}_{\text{increase in time of magnetic energy storage}} + \underbrace{\iiint \vec{E}\cdot\vec{J}\,dV}_{\text{power to heat } (I^2R) \text{ is volume}}$$

surf. area



<u>Sketch of Proof</u>

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\bar{D}}{\partial t}$$

dot $\vec{E}$ into both sides

$$\underbrace{\vec{E}\cdot\nabla\times\vec{H}}_{} = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial\bar{D}}{\partial t}$$

use vector identities and $\nabla\times\vec{E} = -\frac{\partial\bar{B}}{\partial t}$

to get above eqn with left hand $= -\iiint \nabla\cdot(\vec{E}\times\vec{H})\,dV$

which, from divergence theorem, yields the above

4. Write a mathematical expression in both the time-domain and frequency domain for (1) a linearly polarized plane wave and (2), a circularly polarized one.

wave in $+z$ dir.

General:

$$\vec{E} = (a_x \hat{x} + a_y e^{j\delta} \hat{y}) e^{-jkz}$$ (phasor plane wave)

$$\vec{E}(z,t) = a_x \cos(\omega t - kz)\hat{x} + a_y \cos(\omega t - kz + \delta)\vec{y}$$ (time domain)

for linear polarization, $\delta = 0, \pi$
a_x, a_y arbitrary

for circular polarization $\delta = \pm \pi/2$
$a_x = a_y = a$

(1) linearly polarized

$$\boxed{\tilde{\vec{E}}(z) = a_x e^{-jkz} \hat{x}}$$ ← freq. domain

$$\boxed{\vec{E}(z,t) = a_x \cos(\omega t - kz)\hat{x}}$$ ← time domain

one of many (basically 3) possibilities

(2) left circularly polarized

$$\boxed{\tilde{\vec{E}}(z) = (a\hat{x} + a e^{j\pi/2}\hat{y}) e^{-jkz}}$$ ← freq. domain

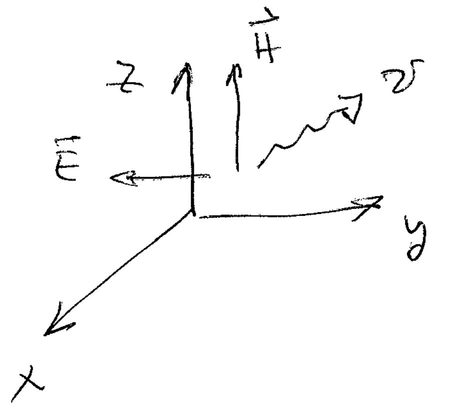
$$\boxed{\vec{E}(z,t) = a\cos(\omega t - kz)\hat{x} - a\sin(\omega t - kz)\vec{y}}$$ ← time domain

one of two possibilities

or equivalently

$$-a\cos(\omega t - kz + \tfrac{\pi}{2})\vec{y}$$

5. A uniform plane wave has a magnetic field intensity $\vec{H} = \vec{z}\ A \sin(dt + bx)$. (1) In what direction is the wave propagating and (2) in what direction is the accompanying electric field?



sin (dt + bx) means going in $-\hat{x}$ direction because to keep phase constant, x must get smaller as time increases

dir. of prop = $\vec{E} \times \vec{H}$, so $\vec{E}$ in $-y$

(1) $\boxed{\text{direction} = -\hat{x}}$

(2) $\boxed{\vec{E} = -\hat{y}}$