1) Derive a wave equation for $\vec{H}$ in free space.
Assume $\vec{J} = \rho = \emptyset$

start by taking the curl of the Maxwell equation $\left[\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}\right]$

(a) $\nabla \times (\nabla \times \vec{H}) = \nabla \times \left(\frac{\partial \vec{D}}{\partial t}\right)$; $\vec{D} = \epsilon_0 \vec{E}$

(b) $\nabla \times (\nabla \times \vec{H}) = \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \vec{E})$; pull out constant ; and partial derivative

Recall Maxwell's equation / Faraday's experiments

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$ use this substitution in equation (b)

c) $\nabla \times (\nabla \times \vec{H}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}$

Now we must use the following identity

$\nabla \times (\nabla \times \vec{H}) = \nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H}$

d) $\underbrace{\nabla(\nabla \cdot \vec{H})}_{\emptyset} - \nabla^2 \vec{H} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}$

no magnetic monopoles
$\nabla \cdot \mu \vec{H} = 0$

(cont.)

Derive a wave equation for $\vec{H}$ in free space (cont.)

e) $$-\nabla^2 \vec{H} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t}$$

or

f) $$\boxed{\nabla^2 \vec{H} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t} = \emptyset}$$

wave equation for $\vec{H}$ in free space

2) Solve the wave equation in Problem #1

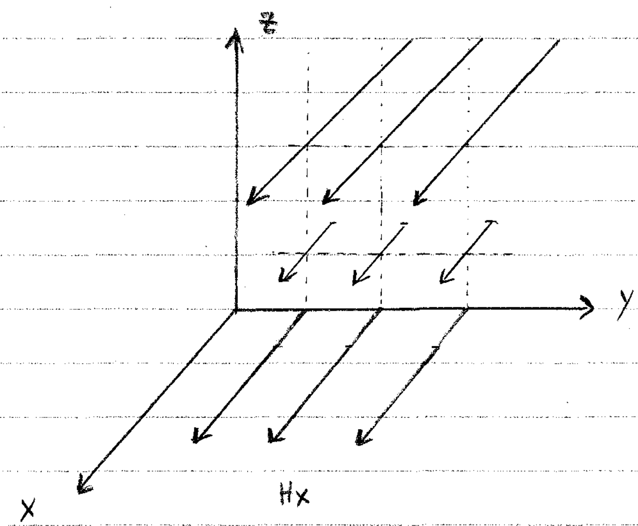
assume: $\vec{H} = H_x(z,t)\,\hat{x} = f(t + z/v)\,\hat{x}$

start with the wave equation that we derived in problem #1.

a) $\nabla^2 \vec{H} - \mu_0 \epsilon_0 \dfrac{\partial^2 \vec{H}}{\partial t} = \emptyset$; wave equation for $\vec{H}$; in free space

b) $\dfrac{\partial^2 \vec{H}}{\partial x^2} + \dfrac{\partial^2 \vec{H}}{\partial y^2} + \dfrac{\partial^2 \vec{H}}{\partial z^2} - \mu_0 \epsilon_0 \dfrac{\partial^2 \vec{H}}{\partial t} = \emptyset$; expanded ; Laplacian

recall, we are assuming $\vec{H} = H_x(z,t)\,\hat{x}$



- Magnitude of $\vec{H}$ varys with z only
- Not a function of x or y

(cont.)

Solve the wave equation in Problem #1 (cont.)

plug $\vec{H}$ into the derived wave equation (eq b)

c) $$\underbrace{\underbrace{\frac{\partial^2}{\partial x^2} H_x(z,t)\hat{x}}_{\emptyset} + \underbrace{\frac{\partial^2}{\partial y^2} H_x(z,t)\hat{x}}_{\emptyset}} + \frac{\partial^2}{\partial z^2} H_x(z,t)\hat{x} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t} H_x(z,t)\hat{x} = \emptyset$$

$H_x(z,t)\hat{x}$ is not a function of x or y

d) $$\frac{\partial^2}{\partial z^2} H_x(z,t)\hat{x} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t} H_x(z,t)\hat{x} = \emptyset$$

this is the wave equation for uniform plane wave

Note: $H_x(z,t)$ = any function of $(t + z/v)$ is a solution where

$$v = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

let $f(t + z/v)$ be any function

PROOF

$$\frac{\partial f(t+z/v)}{\partial z} = \overbrace{\underbrace{\frac{\partial f(t+z/v)}{\partial (t+z/v)}}_{=f'} \frac{\partial (t+z/v)}{\partial z}}^{\text{chain rule}}$$

(cont.)

Solve the wave equation in Problem 1 (cont.)

$$\frac{\partial f(t+z/v)}{\partial z} = f'\left(\frac{1}{c}\right)$$

Now find the second partial derivitive,

$$\frac{\partial^2 f(t+z/v)}{\partial z^2} = \frac{\partial f'}{\partial(t+z/v)}\frac{\partial(t+z/v)}{\partial z} = \frac{\partial^2 f(t+z/v)}{\partial^2(t+z/v)}\left(\frac{1}{c}\right)\left(\frac{1}{c}\right)$$

$$\frac{\partial^2 f(t+z/v)}{\partial z^2} = f''\left(\frac{1}{c^2}\right)$$; this is the first ; term in the original ; equation (d)

Now lets work with the second term of equation (d)

$$\frac{\partial f}{\partial t} = \underbrace{\frac{\partial f(t+z/v)}{\partial f(t+z/v)}}_{f'} \times \underbrace{\frac{\partial(t+z/v)}{\partial t}}_{1} = f'$$

∴ the second partial derivite is simply

$$\frac{\partial^2 f}{\partial t^2} = f''$$

Now plug in all derivitives into equation (d)

e) $$\frac{1}{c^2} f''\left(t+\frac{z}{v}\right) - \mu_0 \epsilon_0 f'' = \phi$$

(cont.)

Solve the wave equation in Problem 1 (cont.)

if $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ then equation is solved

Proved 2 things,

1) any function, $f(t + z/v)$ is a solution to wave equation

2) Speed of wave $= \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

We know everything about $\vec{H}$, lets investigate $\vec{E}$. Start with Maxwell's equation along with Ampere's experiments.

f) $\underbrace{\nabla \times \vec{H}} = \vec{J} + \frac{\partial D}{\partial t}$; but $\vec{J} = \emptyset$

$$\nabla \times \vec{H} = \nabla \times (H_x(z,t)\hat{x}) = \hat{x}\left(\underbrace{\frac{\overset{0}{\partial H_z}}{\partial y}}_{0} - \underbrace{\frac{\overset{0}{\partial H_y}}{\partial z}}_{0}\right) + \hat{y}\left(\frac{\partial H_x}{\partial z} - \underbrace{\frac{\overset{0}{\partial H_z}}{\partial x}}_{0}\right) + \hat{z}\left(\underbrace{\frac{\overset{0}{\partial H_y}}{\partial x}}_{0} - \underbrace{\frac{\partial H_x}{\partial y}}_{0}\right)$$

g) $\hat{y}\left(\frac{\partial H_x(z,t)}{\partial z}\right) = \frac{\partial \vec{D}}{\partial t} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Solve for $\vec{E}$,

(cont.)

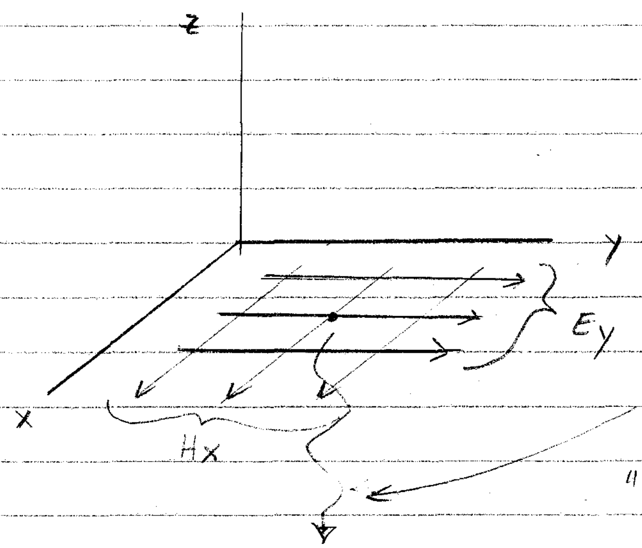
Solve the wave equation in Problem 1 (cont.)

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon_0}\,\frac{\partial H_x(z,t)}{\partial z}$$

$$\int_0^t \frac{\partial \vec{E}}{\partial t}\,dt = \frac{1}{\epsilon_0}\int_0^t \underbrace{\frac{\partial H_x(z,t)}{\partial z}}_{F'\left(\frac{1}{c}\right)}\,dt$$

$$\vec{E} = \frac{1}{\epsilon_0} \times \sqrt{\mu_0 \epsilon_0}\int_0^t f'\,dt$$

$$\vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}}\int_0^t \frac{\partial f}{\partial t}\,dt = \sqrt{\frac{\mu_0}{\epsilon_0}}\,H_x\,\hat{y}$$



propogating in the
$-z$ direction
"direction of power flow"

$$\vec{S} = \vec{E} \times \vec{H}$$
$$-\hat{z} = \hat{y} \times \hat{x}$$

$$c = \frac{E_y}{B_x} = \frac{E_y}{\mu_0 H_x} = \frac{1}{\mu_0}\sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

direction is $-z$ and velocity $= c$

3) Derive a wave equation for $\vec{E}$ in free space where $\rho = 0$ and $\vec{J}(\vec{r},t) = \sigma\vec{E}(\vec{r},t)$

start with Maxwell's equation | Faraday experiments

a) $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial}{\partial t}\vec{H}$; now take the curl

b) $\nabla\times(\nabla\times\vec{E}) = -\mu_0\frac{\partial}{\partial t}(\nabla\times\vec{H})$; but $\nabla\times\vec{H} = \vec{J} + \frac{\partial D}{\partial t}$

c) $\nabla\times(\nabla\times\vec{E}) = -\mu_0\frac{\partial}{\partial t}\left[\vec{J} + \frac{\partial D}{\partial t}\right]$; but $\vec{D} = \epsilon_0\vec{E}$; and $\vec{J} = \sigma\vec{E}$

d) $\nabla\times(\nabla\times\vec{E}) = -\mu_0\frac{\partial}{\partial t}\left[\sigma\vec{E} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]$

Identity: $\nabla\times(\nabla\times\vec{E}) = \nabla\underbrace{(\nabla\cdot\vec{E})}_{\emptyset} - \nabla^2\vec{E}$

e) $-\nabla^2\vec{E} = -\mu_0\frac{\partial}{\partial t}\left[\sigma\vec{E} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]$

f) $$\boxed{\nabla^2\vec{E} - \mu_0\sigma\frac{\partial\vec{E}}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = \emptyset}$$