

$$\vec{\mathcal{J}} = \nabla \times \vec{\mathcal{H}} - \frac{\partial \vec{\mathcal{D}}}{\partial t}$$

and take the dot product of both sides with $\vec{\mathcal{E}}$ to obtain $\vec{\mathcal{E}} \cdot \vec{\mathcal{J}}$ in units of W-m⁻³:

$$\vec{\mathcal{E}} \cdot \vec{\mathcal{J}} = \vec{\mathcal{E}} \cdot (\nabla \times \vec{\mathcal{H}}) - \vec{\mathcal{E}} \cdot \frac{\partial \vec{\mathcal{D}}}{\partial t}$$

and use the vector identity

$$\nabla \cdot (\vec{\mathcal{E}} \times \vec{\mathcal{H}}) = \vec{\mathcal{H}} \cdot (\nabla \times \vec{\mathcal{E}}) - \vec{\mathcal{E}} \cdot (\nabla \times \vec{\mathcal{H}})$$

to find

$$\vec{\mathcal{E}} \cdot \vec{\mathcal{J}} = \vec{\mathcal{H}} \cdot (\nabla \times \vec{\mathcal{E}}) - \nabla \cdot (\vec{\mathcal{E}} \times \vec{\mathcal{H}}) - \vec{\mathcal{E}} \cdot \frac{\partial \vec{\mathcal{D}}}{\partial t} \quad [8.27]$$

We now use [8.1a],

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}$$

and substitute in [8.27] to obtain

$$\vec{\mathcal{E}} \cdot \vec{\mathcal{J}} = -\vec{\mathcal{H}} \cdot \frac{\partial \vec{\mathcal{B}}}{\partial t} - \vec{\mathcal{E}} \cdot \frac{\partial \vec{\mathcal{D}}}{\partial t} - \nabla \cdot (\vec{\mathcal{E}} \times \vec{\mathcal{H}}) \quad [8.28]$$

We now consider the two terms with the time derivatives. For a simple medium²⁹ (i.e., ϵ and μ simple constants) we can write

$$\vec{\mathcal{H}} \cdot \frac{\partial \vec{\mathcal{B}}}{\partial t} = \vec{\mathcal{H}} \cdot \frac{\partial (\mu \vec{\mathcal{H}})}{\partial t} = \frac{1}{2} \frac{\partial (\mu \vec{\mathcal{H}} \cdot \vec{\mathcal{H}})}{\partial t} = \frac{\partial}{\partial t} \left(\frac{1}{2} \mu |\vec{\mathcal{H}}|^2 \right)$$

and

$$\vec{\mathcal{E}} \cdot \frac{\partial \vec{\mathcal{D}}}{\partial t} = \vec{\mathcal{E}} \cdot \frac{\partial (\epsilon \vec{\mathcal{E}})}{\partial t} = \frac{1}{2} \frac{\partial (\epsilon \vec{\mathcal{E}} \cdot \vec{\mathcal{E}})}{\partial t} = \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon |\vec{\mathcal{E}}|^2 \right)$$

respectively. With these substitutions, [8.28] can be rewritten as

$$\vec{\mathcal{E}} \cdot \vec{\mathcal{J}} = -\frac{\partial}{\partial t} \left(\frac{1}{2} \mu |\vec{\mathcal{H}}|^2 \right) - \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon |\vec{\mathcal{E}}|^2 \right) - \nabla \cdot (\vec{\mathcal{E}} \times \vec{\mathcal{H}}) \quad [8.29]$$

²⁹The more general form of Poynting's theorem, without these assumptions about the medium, is

$$\int_V \vec{\mathcal{E}} \cdot \vec{\mathcal{J}} dv = -\frac{\partial}{\partial t} \int_V \left(\vec{\mathcal{H}} \cdot \frac{\partial \vec{\mathcal{B}}}{\partial t} + \vec{\mathcal{E}} \cdot \frac{\partial \vec{\mathcal{D}}}{\partial t} \right) dv - \oint_S (\vec{\mathcal{E}} \times \vec{\mathcal{H}}) \cdot d\vec{s}$$

which is valid even for nonlinear media, as long as there are no hysteresis effects. For ferromagnetic materials, for which the relation between $\vec{\mathcal{B}}$ and $\vec{\mathcal{H}}$ is often multivalued due to hysteresis, an additional amount of energy is deposited within the material.

$$-\oint (\vec{\mathcal{E}} \times \vec{\mathcal{H}}) \cdot d\vec{s} = \iiint \vec{\mathcal{E}} \cdot \vec{\mathcal{J}} dv + \frac{\partial}{\partial t} \iiint \left(\frac{1}{2} \epsilon |\vec{\mathcal{E}}|^2 + \frac{1}{2} \mu |\vec{\mathcal{H}}|^2 \right) dv$$

Integrating [8.29] over an arbitrary volume V we have

$$\int_V \bar{\mathcal{E}} \cdot \bar{\mathcal{J}} dv = -\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \mu |\bar{\mathcal{H}}|^2 + \frac{1}{2} \epsilon |\bar{\mathcal{E}}|^2 \right) dv - \int_V \nabla \cdot (\bar{\mathcal{E}} \times \bar{\mathcal{H}}) dv \quad [8.30]$$

Using the divergence theorem on the last term in [8.30] we find

$$\int_V \bar{\mathcal{E}} \cdot \bar{\mathcal{J}} dv = -\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \mu |\bar{\mathcal{H}}|^2 + \frac{1}{2} \epsilon |\bar{\mathcal{E}}|^2 \right) dv - \oint_S (\bar{\mathcal{E}} \times \bar{\mathcal{H}}) \cdot ds \quad [8.31]$$

where the surface S encloses the volume V .

We can now interpret the various terms in [8.31] physically. The left-hand term is the generalization of Joule's law and represents the instantaneous power dissipated in the volume V . If $\bar{\mathcal{E}}$ is the electric field that produces $\bar{\mathcal{J}}$ in a lossy medium, this term represents the ohmic (I^2R) power loss in the medium. Note that in a simple isotropic medium (i.e., σ a simple constant), $\bar{\mathcal{E}}$ and $\bar{\mathcal{J}}$ are in the same direction. However, in general this is not true. In Earth's ionosphere (which is an anisotropic medium because of the presence of Earth's magnetic field), for example, an applied electric field in one direction can cause current flow in that direction as well as in other directions. Even in such cases, however, $\bar{\mathcal{E}} \cdot \bar{\mathcal{J}}$ still represents the power dissipated per unit volume, although $\bar{\mathcal{J}}$ and $\bar{\mathcal{E}}$ are not parallel. Alternatively, there could be an energy source within the volume V , such as an antenna carrying current, in which case $\bar{\mathcal{E}} \cdot \bar{\mathcal{J}}$ is negative and represents power flow out of that region.

The first term on the right-hand side represents the rate at which the electromagnetic energy stored in volume V decreases (negative sign), with the terms $\frac{1}{2} \mu |\bar{\mathcal{H}}|^2$ and $\frac{1}{2} \epsilon |\bar{\mathcal{E}}|^2$ representing, respectively, the magnetic and electric energy densities. Note that, strictly speaking, the quantities $W_e = \frac{1}{2} \epsilon |\bar{\mathcal{E}}|^2$ and $W_m = \frac{1}{2} \mu |\bar{\mathcal{H}}|^2$ are known to represent electric and magnetic energy densities for static fields. However, it is generally assumed that these quantities also represent stored energy densities in the case of time-varying fields.³⁰

From conservation of energy, the last term in [8.31] must represent the flow of energy inward or outward through the surface S enclosing the volume V . Thus, the vector $\bar{\mathcal{P}} = \bar{\mathcal{E}} \times \bar{\mathcal{H}}$, which has dimensions of watts per square meter ($\text{W}\cdot\text{m}^{-2}$), appears to be a measure of the rate of energy flow per unit area at any point on the surface S . The direction of power flow is perpendicular to both $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$. In other words, the power density in an electromagnetic wave is given by

³⁰Such an assumption is entirely reasonable since the energy density is defined at a given point. From another point of view, we can consider Poynting's theorem or [8.31] to be the definition of energy density for time-varying fields. The correct amount of total electromagnetic energy is always obtained by assigning an amount $\frac{1}{2}(\bar{\mathcal{D}} \cdot \bar{\mathcal{H}} + \bar{\mathcal{D}} \cdot \bar{\mathcal{E}}) = \frac{1}{2}(\epsilon |\bar{\mathcal{E}}|^2 + \mu |\bar{\mathcal{H}}|^2)$ to each unit of volume. Other previously cited expressions for static energy densities, such as $\frac{1}{2} \rho \Phi$ for electrostatic fields, are not applicable for time-varying fields. See Chapter II-27 of R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison-Wesley, 1964, and Section 2.19 of J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, 1941.