SOLUTIONS: HOUR TEST #2, MARCH 6, 2009

Problem 1.
Write down the point (differential) form of Gauss's Law (which is one of Maxwell's equations) and derive the integral form of Gauss's Law starting with the point form. Show all steps in the derivation. Do both the point and the integral forms you have written contain the same information? If not, why not?

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point form of Gauss' Law $\nabla \cdot \vec{D} = \rho_v$ (1)

Integrate both sides over a volume

$$\iiint_{vol} \nabla \cdot \vec{D}\, dV = \iiint_{vol} \rho_v\, dV \quad (2)$$

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Apply divergence theorem

$$\iiint \nabla \cdot \vec{D}\, dV = \oiint \vec{D} \cdot d\vec{s} \quad (3)$$

surface enclosing the volume integrated over

to (2)

Integral form of Gauss' Law $\Rightarrow$ $\oiint \vec{D} \cdot d\vec{s} = \iiint \rho_v\, dV = Q_{enclosed}$ (4)

(volume charge density)

In (4) $Q_{enclosed}$ is only from ρ_v, whereas

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in general $Q_{enclosed}$ can include point charges and line charges

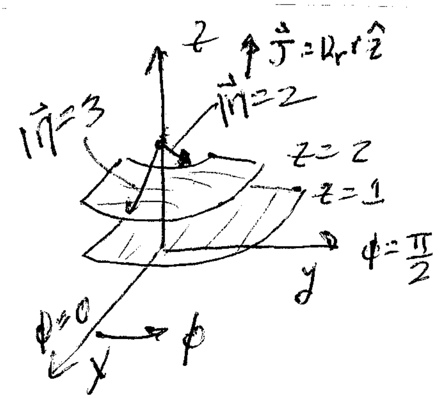
Problem 2.

Given that a current density is $\vec{J} = K_1 r\hat{z}\ A/m^2$ (in cylindrical coordinates) and is formed from a uniform collection of positive charged particles moving at constant velocity $K_2\hat{z}\ m/s$, where K_1 and K_2 are constants, find:

(1) the total electrical charge contained between z = 1 and z = 2, φ = 0 and φ = 90°, r = 2, and r = 3, and

(2) the total current through a plane surface at z = 2 bounded by φ = 0 and φ = 90°, and r = 2 and r = 3.

(1) Since $\vec{J} = \rho_v \vec{u}$ and $\vec{u} = K_2 \hat{z}$ m/s



$$\vec{J} = K_1 r \hat{z} = K_2 \rho_v \hat{z}$$

$$\therefore \rho_v = \frac{K_1 r}{K_2} \quad C/m^3$$

$$Q = \int_{z=1}^{2}\int_{r=2}^{3}\int_{\phi=0}^{\pi/2} \underbrace{\frac{K_1 r}{K_2}}_{\rho_v} \underbrace{r\,dr\,d\phi\,dz}_{dV} = \int_{z=1}^{2}\int_{r=2}^{3}\int_{\phi=0}^{2\pi} \frac{K_1}{K_2} r^2\,dr\,d\phi\,dz$$

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$$Q = \underbrace{\left(\frac{\pi}{2} - 0\right)}_{\phi}\underbrace{(2-1)}_{z} \frac{K_1}{K_2} \frac{r^3}{3}\Big|_2^3 = \frac{\pi}{2}\frac{K_1}{K_2}\left(\frac{27}{3} - \frac{8}{3}\right) = \frac{\pi}{2}\frac{K_1}{K_2}\frac{19}{3}$$

$$Q = \frac{19}{6}\pi \frac{K_1}{K_2}\ C$$

(2) $$I = \int_{r=2}^{3}\int_{\phi=0}^{\pi/2} \vec{J}\cdot\vec{ds}\Big|_{\text{at } z=2} = \int_{r=2}^{3}\int_{\phi=0}^{\pi/2} K_1 r\hat{z} \cdot \underbrace{r\,dr\,d\phi\,\hat{z}}_{\vec{ds}} = K_1 \frac{\pi}{2} \frac{r^3}{3}\Big|_2^3 = K_1 \frac{19}{6}\pi \text{ Amps}$$

$Q_{TOT} = \frac{19}{6}\pi \frac{K_1}{K_2}$ Coul
$I_{TOT} = \frac{19}{6}\pi K_1$ Amps

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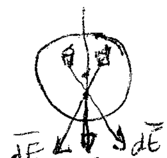
Problem 3.

A charged spherical volume of radius "b", whose center is at the origin of a spherical coordinate system contains a non-uniform volume charge density $\rho_v = KR^{-1}\,C/m^3$ where K is a constant. Draw a picture of the problem. Find and plot the electric field intensity at $R \le b$.



(1) Argue that $\vec{E}$ must be in $\hat{R}$ direction and constant on a spherical Gaussian surface centered at the origin (shown in red)

$dV = R^2 \sin\theta \, d\theta d\phi dR$

(2) for $|\vec{R}| \le b$ integrate over a Gaussian sphere and set result equal to charge inside sphere

$$\oiint \vec{D}\cdot d\vec{S}\Big|_R = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{R'=0}^{R\le b}\left(\frac{K}{R'}\right) \underbrace{R'^2 \sin\theta \, d\theta \, d\phi \, dR'}_{dV}$$

$\epsilon E_R(\vec{R})\hat{R}$ $\quad R^2 \sin\theta \, d\theta \, d\phi \, \hat{R}$ $\quad \rho_v$

$$\epsilon E_R \, 4\pi R^2 \Big|_{|\vec{R}|\le b} = 4\pi K \int_{R'=0}^{R<b} R' dR' = \frac{4\pi K R^2}{2}\Big|_{|\vec{R}|\le b}$$

$$E_R = \frac{K}{2\epsilon} \quad V/m$$

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or

$$\frac{1}{R^2}\frac{\partial}{\partial R} R^2 D_R = \frac{K}{R} \qquad (\nabla\cdot\vec{D} = \rho_v)$$

$$\frac{\partial}{\partial r}(R^2 D_R) = KR$$

$$\int_0^{R'} d(R^2 D_R) = \int_0^{R'} KR \, dR$$

$$R'^2 D_{R'} = \frac{K R'^2}{2}$$

$$D_R = \frac{K}{2}, \quad E_R = \frac{K}{2\epsilon}$$

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$$\vec{E} = \frac{K}{2\epsilon}\hat{R} \quad V/m$$

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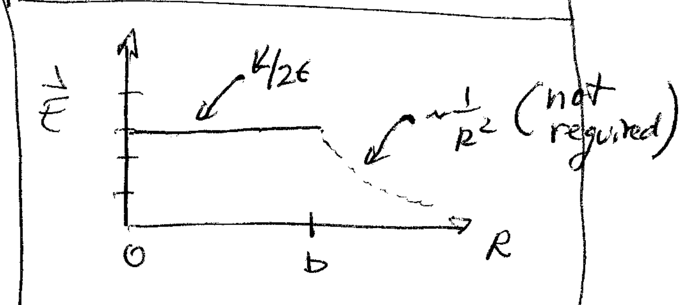
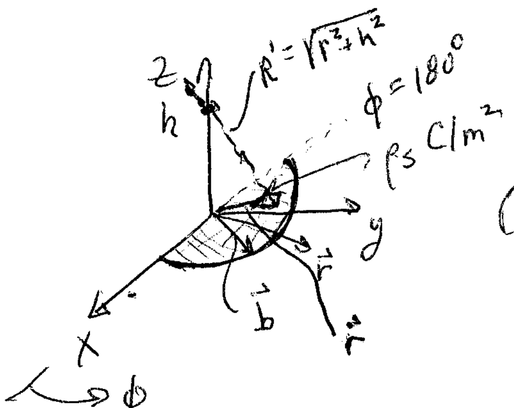


Problem 4.

A uniformly-charged half-disk of radius "b" (ρ_s = constant, from $\varphi = 0$ to $\varphi = 180°$) is located in the z = 0 plane, as shown. Set up an expression that can be integrated to find the electric field intensity at the point r = 0, z = h. In what vector direction is the electric field intensity at z = h?



$$\vec{dE}(z=h) = \overbrace{\frac{\rho_s\, r\,dr\,d\phi}{4\pi\epsilon(r^2+h^2)}}^{dQ}\left(\frac{-r\hat{r}+h\hat{z}}{\sqrt{r^2+h^2}}\right)$$

$$\vec{dE} = \frac{dQ}{4\pi\epsilon R'^2}\hat{R}'$$

$R'^2 = r^2+h^2$

magnitude of dE at $z=h$, $r=0$

unit vector $\hat{R}'$ at $z=h$, $r=0$

since $\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$

$$\vec{E}(z=h) = \int_{r=0}^{b}\int_{\phi=0}^{\pi} \frac{\rho_s}{4\pi\epsilon}\left(\frac{-r\cos\phi\,\hat{x} - r\sin\phi\,\hat{y} + h\hat{z}}{(r^2+h^2)^{3/2}}\right) r\,dr\,d\phi$$

the x-component is zero, by integration or examination of the symmetry of the problem

$$\vec{E}(z=h, r=0) = \int_{r=0}^{b}\int_{\phi=0}^{\pi} \frac{\rho_s\,[-r\sin\phi\,\hat{y} + h\hat{z}]\,r\,dr\,d\phi}{4\pi\epsilon\,(r^2+h^2)^{3/2}}$$

+15	$\vec{E}(r=0, z=h) =$ ↓ V/m
+10	Direction of field at $r=0, z=h$ is $-\hat{y}$ and $+\hat{z}$