FINAL EXAM, EEL 3472, SPRING 2009

SOLUTIONS

1. An infinitely-long co-axial transmission line is oriented along the z-axis. The outer radius of the cylindrical inner conductor is at $r = a$; the inner radius of the outer cylindrical conductor is at $r = b$. From a solution of Maxwell's equations, the electric field intensity between $r = a$ and $r = b$ is found to be given by

$$\vec{E}(r,z,t) = \frac{k}{r}\cos(2\pi x 10^{6} t + 2\pi x 10^{-2} z)\hat{r} \quad \text{where } k \text{ is a constant.}$$

The magnetic field intensity is given by $\vec{H}(r,z,t) = -\left|\vec{E}\right|\sqrt{\frac{\varepsilon}{\mu}}\,\hat{\phi}$.

25 points

(a) What is the direction of wave propagation?
(b) What is the wave propagation speed?
(c) What is the voltage between the inner and outer conductor?
(d) What is the current on the inner and outer conductors?
(e) What is the characteristic impedance of the transmission line?

Show all work and list all assumptions.

(a) $E_r = \frac{K}{r}\cos(2\pi\times10^{6}t + 2\pi\times10^{-2}z)$ V/m

$H_\phi = \dfrac{-K\sqrt{\frac{\varepsilon}{\mu}}}{r}\underbrace{\cos(2\pi\times10^{6}t + 2\pi\times10^{-2}z)}$ A/m

to keep phase constant, as t increases z must decrease, so wave goes in $-\hat{z}$ dir

(b) of form $\cos(\omega t + kz)$, $u_p = \frac{\omega}{k} = \frac{2\pi\times10^{6}}{2\pi\times10^{-2}} = 10^{8}$ m/s

(c) $|V_{ba}| = \left|-\int_a^b \vec{E}\cdot\vec{dl}\right| = \left|-\int_a^b E_r dr\right| = K\cos(\;)\int_a^b\frac{dr}{r} = K\cos(\;)\ln b/a$

Voltage diff. can be defined because in any $z =$ const plane $\nabla\times\vec{E} = 0$, since there is no $\frac{\partial\vec{B}}{\partial t}$ in the z-direction (only B_ϕ), and hence $\oint\vec{E}\cdot\vec{dl} = 0$ in any z = const. plane & defined voltage is independent of path from $r = a$ to $r = b$

(d) $\oint\vec{H}\cdot\vec{dl} = I_{encl}$

note: there is no $\iint\frac{\partial\vec{D}}{\partial t}\cdot\vec{dS}$ enclosed because no D_z

path of $\oint H\cdot\vec{dl}$

$H_\phi(r)\,2\pi r = I_{encl}$

$-\dfrac{K\sqrt{\frac{\varepsilon}{\mu}}\cos(\;)}{\cancel{r}}2\pi\cancel{r} = I_{encl}$

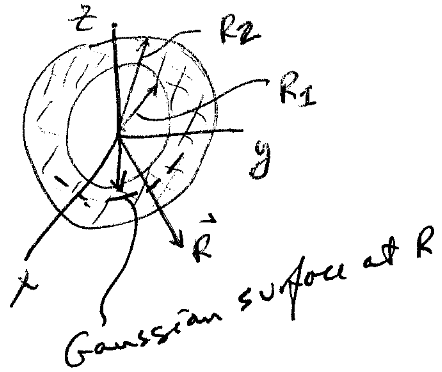
$|I_{encl}| = K\sqrt{\frac{\varepsilon}{\mu}}\,2\pi\cos(\;)$

(e) $Z = \dfrac{|V^{+}|}{|I^{+}|}$ (or $-$) $= \dfrac{K\cos(\;)\ln b/a}{K\sqrt{\frac{\varepsilon}{\mu}}\,2\pi\cos(\;)}$

$= \sqrt{\frac{\mu}{\varepsilon}}\frac{\ln b/a}{2\pi}$

(a) $(-\hat{z})$ direction
(b) $u_p = 10^{8}$ m/s
(c) $K\cos(\;)\ln b/a$ Volts
(d) $K\sqrt{\frac{\varepsilon}{\mu}}\,2\pi\cos(\;)$ Amps
(e) $Z = \sqrt{\frac{\mu}{\varepsilon}}\frac{\ln b/a}{2\pi}$ Ohms

2. A volume charge density $\rho_v = kR^{-2}\ C/m^3$, in a spherical coordinate system, is located between $R = R_1$ and $R = R_2$ (in a spherical annulus). The charge density is zero from $R = 0$ to $R = R_1$. Find and plot the electric field intensity from $R = 0$ to $R = R_2$. Show all work and list all assumptions. 20 points



$$\oint\!\!\oint \bar{D}\cdot d\bar{s} = Q_{enclosed}$$

from symmetry $\vec{D} = D_R(R)\,\hat{R}$

– on Gaussian surf. at R

$$\underbrace{D_R(R)\,4\pi R^2}_{\text{G. sphere area}} = \underbrace{\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{R_1}^{R} \overbrace{kR^{-2}}^{\rho_v}\ \overbrace{R^2\sin\theta\,d\theta\,d\phi\,dR}^{dV}}_{Q_{encl.}\text{ inside } R}$$

$$D_R(R)\,\cancel{4\pi}R^2 = \cancel{4\pi}\,K\,(R-R_1)$$

$$D_R(R) = K\,\frac{R-R_1}{R^2} = K\left(\frac{1}{R} - \frac{R_1}{R^2}\right)$$

$$E_R(R) = \frac{K}{\epsilon}\left(\frac{1}{R} - \frac{R_1}{R^2}\right), \quad R \ge R_1,\ R \le R_2$$

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for $R \le R_1$

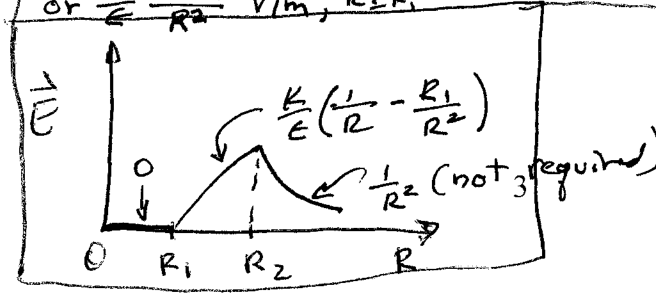
Gaussian sphere in hole

$$D_R(R)\,4\pi R^2 = 0 \quad \text{(no charge enclosed)}$$

$$E_R(R) = 0, \quad R \le R_1$$

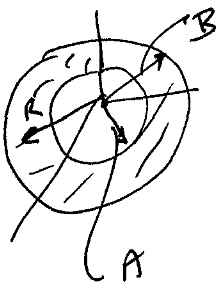
$$\boxed{\vec{E} = \begin{cases} 0, & R \le R_1 \quad V/m \\ \dfrac{K}{\epsilon}\left(\dfrac{1}{R} - \dfrac{R_1}{R^2}\right), & R \ge R_1,\ R \le R_2 \end{cases}}$$

or $\frac{K}{\epsilon}\frac{(R-R_1)}{R^2}$ V/m, $R \ge R_1$



3. Two concentric perfectly-conducting spheres of radii A and B (B>A) are separated by a material of conductivity $\sigma = KR^{-2}$ where K is a constant. Find the electric field intensity between the spheres if a current "I" flows from one to the other. Find the resistance between the spheres. Show all work and list all assumptions.

20 points



$\nabla \cdot \vec{J} = 0$, assume only $J_R(R)$ from symmetry

$\frac{1}{R^2}\frac{\partial}{\partial r} R^2 J_R(R) = 0$

$R^2 J_R(R) = K$ (constant)

$J_R = \frac{K}{R^2}$

$I = \iint \bar{J} \cdot d\bar{s}$

Since $I = J_R(R) 4\pi R^2$ at any R

$J_R(R) = \frac{I}{4\pi R^2}$, that is, $K = \frac{I}{4\pi}$

$\vec{J} = \sigma \vec{E}$

so $\vec{E} = \frac{\vec{J}}{\sigma} = \frac{I \hat{R}}{4\pi R^2 K R^{-2}} = \frac{I}{4\pi K}\hat{R}$

$R = \frac{V}{I} = \frac{\left| \int_{R=A}^{B} \bar{E} \cdot d\bar{l} \right|}{I}$

$V = \int_A^B \frac{I}{4\pi K} dR = \frac{I(B-A)}{4\pi K}$

$R = \frac{V}{I} = \frac{B-A}{4\pi K}$

$\vec{E} = \frac{I}{4\pi K}\hat{R}$ V/m
$R = \frac{(B-A)}{4\pi K}$ Ohms

20 points

4. What expression can be used to convert Maxwell's two curl equations from point to integral form? Give an example of its use. What expression can be used to convert Maxwell's two divergence equations from point to integral form? Give an example of its use.

Convert curl eqns with Stoke's Thm

$$\iint (\nabla \times \bar{A}) \cdot d\vec{s} = \oint \bar{A} \cdot d\vec{l}$$

pt form: $\nabla \times \vec{E} = -\frac{\partial \bar{B}}{\partial t}$, dot each side into $d\vec{s}$ & integrate over an open surface

$$\underbrace{\iint (\nabla \times \vec{E}) \cdot d\bar{s}}_{\text{apply Stoke's Th}} = -\iint \frac{\partial \bar{B}}{\partial t} \cdot ds$$

⇓

int. form: $\oint \bar{E} \cdot d\bar{l} = -\iint \frac{\partial \bar{B}}{\partial t} \cdot d\vec{s}$

Convert divergence eqns with Divergence Thm

$$\iiint \nabla \cdot \vec{A} \, dV = \oiint \bar{A} \cdot d\vec{s}$$

pt. form: $\nabla \cdot \vec{D} = \rho_v$, integrate over volume

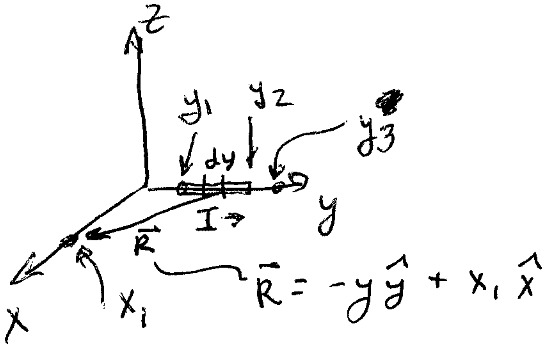
$$\underbrace{\iiint \nabla \cdot \bar{D} \, dv}_{\text{use div Thm}} = \iiint \rho_v \, dv$$

⇓

int. form: $\oiint \bar{D} \cdot d\bar{s} = Q_{encl}$

20 points

5. Current flows along the y-axis from $y = y_1$ to $y = y_2$, as shown. Set up an expression which can be integrated (do not integrate) for the magnetic field intensity on the x-axis at $x = x_1$. What is the magnetic field on the y-axis at $y = y_3 > y_2$? Draw a picture and show all steps.



$$d\vec{H} = \frac{I}{4\pi}\frac{d\vec{l}\times\hat{R}}{R^2}$$

$$d\vec{l} = dy\,\hat{y}$$

$$\vec{R} = -y\hat{y} + x_1\hat{x}, \quad \hat{R} = \frac{-y\hat{y} + x_1\hat{x}}{\sqrt{y^2 + x_1^2}}$$

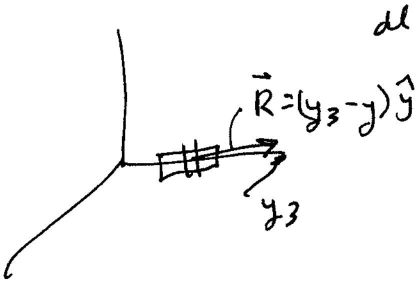
$$|\vec{R}|^2 = y^2 + x_1^2$$

$$d\vec{H}(x_1,0,0) = \frac{I}{4\pi}\frac{\hat{y}\,dy \times (x_1\hat{x} - y\hat{y})}{(y^2 + x_1^2)^{3/2}}$$

$$\int_{y_1}^{y_2} d\vec{H}(x_1,0,0) = -\hat{z}\frac{I x_1}{4\pi}\int_{y_1}^{y_2}\frac{dy}{(x_1^2+y^2)^{3/2}}$$

at $y = y^3$

$$d\vec{l}\times\hat{R} = y\hat{y} \times \frac{(y_3 - y)\hat{y}}{(y_3 - y)} = 0 \qquad (\hat{y}\times\hat{y} = 0)$$



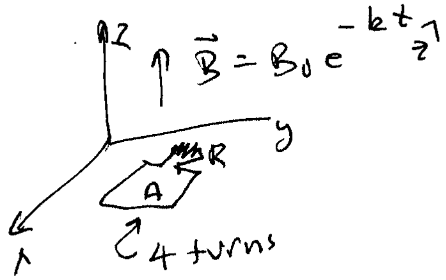
no field along y-axis

$$\boxed{\vec{H}(x_1,0,0) = -\hat{z}\frac{I x_1}{4\pi}\int_{y_1}^{y_2}\frac{dy}{(y^2+x_1^2)^{3/2}}}$$

$$\boxed{\vec{H}(0,y_3,0) = 0 \quad A/m}$$

20 points

6. A magnetic flux density $\vec{B} = B_o e^{-kt}\ \hat{z}\ Wb/m^2$, where t is time and B_o and k are constants, exists in the presence of a 4-turn loop of wire, each turn having the same area A, whose plane is in the x-y plane. The 4 turns are terminated by a physical resistance R, and the total resistance of the wire of the 4-turn loop is R_{wire}. Find the voltage across the physical resistor. Draw a picture and show all steps.



$$V_{emf} = -N \iint \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$= -4A \frac{\partial}{\partial t} B_0 e^{-kt}$$

$$= 4kAB_0 e^{-kt}$$

$$V_{phys.\ resistor} = V_{emf} \frac{R}{R + R_{wire}}$$

$$= 4kAB_0 e^{-kt} \frac{R}{R + R_{wire}}$$

$$\boxed{V = 4kAB_0 e^{-kt} \frac{R}{R + R_{wire}}} \text{ volts}$$

+15 for [illegible] voltage, not division