EEL 3472 HW #5 SOLUTIONS due Friday Feb. 20, 2009 (1)

Problem 3.1 Vector **A** starts at point $(1,-1,-3)$ and ends at point $(2,-1,0)$. Find a unit vector in the direction of **A**.

Solution:

$$\mathbf{A} = \hat{\mathbf{x}}(2-1)+\hat{\mathbf{y}}(-1-(-1))+\hat{\mathbf{z}}(0-(-3)) = \hat{\mathbf{x}}+\hat{\mathbf{z}}3,$$
$$|\mathbf{A}| = \sqrt{1+9} = 3.16,$$
$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}}+\hat{\mathbf{z}}3}{3.16} = \hat{\mathbf{x}}0.32+\hat{\mathbf{z}}0.95.$$

Problem 3.2 Given vectors $\mathbf{A} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}}$, $\mathbf{B} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}+\hat{\mathbf{z}}3$, and $\mathbf{C} = \hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2$, show that **C** is perpendicular to both **A** and **B**.

$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\underbrace{\cos\theta}_{\text{zero if } \theta = 90^\circ}$

Solution:

$$\mathbf{A}\cdot\mathbf{C} = (\hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}})\cdot(\hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2) = 8-6-2 = 0,$$
$$\mathbf{B}\cdot\mathbf{C} = (\hat{\mathbf{x}}2-\hat{\mathbf{y}}+\hat{\mathbf{z}}3)\cdot(\hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2) = 8-2-6 = 0.$$

Problem 3.5 Given vectors $\mathbf{A} = \hat{\mathbf{x}}+\hat{\mathbf{y}}2-\hat{\mathbf{z}}3$, $\mathbf{B} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}4$, and $\mathbf{C} = \hat{\mathbf{y}}2-\hat{\mathbf{z}}4$, find

(a) A and $\hat{\mathbf{a}}$,
(b) the component of **B** along **C**,
(c) θ_{AC},
(d) $\mathbf{A}\times\mathbf{C}$,
(e) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$,
(f) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$,
(g) $\hat{\mathbf{x}}\times\mathbf{B}$, and
(h) $(\mathbf{A}\times\hat{\mathbf{y}})\cdot\hat{\mathbf{z}}$.

Solution:

(a) From Eq. (3.4),

$$A = \sqrt{1^2+2^2+(-3)^2} = \sqrt{14},$$

and, from Eq. (3.5),

$$\hat{\mathbf{a}}_A = \frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}2-\hat{\mathbf{z}}3}{\sqrt{14}}.$$

(b) The component of **B** along **C** (see Section 3-1.4) is given by

$$B\cos\theta_{BC} = \frac{\mathbf{B}\cdot\mathbf{C}}{C} = \frac{-8}{\sqrt{20}} = -1.8.$$

$\vec{B}\cdot\vec{C} = |\vec{B}||\vec{C}|\cos\theta_{BC}$

(c) From Eq. (3.21),

$$\theta_{AC} = \cos^{-1}\frac{\mathbf{A}\cdot\mathbf{C}}{AC} = \cos^{-1}\frac{4+12}{\sqrt{14}\sqrt{20}} = \cos^{-1}\frac{16}{\sqrt{280}} = 17.0^\circ.$$

(d) From Eq. (3.27),

$$\mathbf{A}\times\mathbf{C} = \hat{\mathbf{x}}(2(-4)-(-3)2)+\hat{\mathbf{y}}((-3)0-1(-4))+\hat{\mathbf{z}}(1(2)-2(0)) = -\hat{\mathbf{x}}2+\hat{\mathbf{y}}4+\hat{\mathbf{z}}2.$$

(e) From Eq. (3.27) and Eq. (3.17),

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot(\hat{\mathbf{x}}16+\hat{\mathbf{y}}8+\hat{\mathbf{z}}4) = 1(16)+2(8)+(-3)4 = 20.$$

Eq. (3.30) could also have been used in the solution. Also, Eq. (3.29) could be used in conjunction with the result of part (d).

(f) By repeated application of Eq. (3.27),

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\times(\hat{\mathbf{x}}16+\hat{\mathbf{y}}8+\hat{\mathbf{z}}4) = \hat{\mathbf{x}}32-\hat{\mathbf{y}}52-\hat{\mathbf{z}}24.$$

Eq. (3.33) could also have been used.

(g) From Eq. (3.27),

$$\hat{\mathbf{x}}\times\mathbf{B} = -\hat{\mathbf{z}}4.$$

(h) From Eq. (3.27) and Eq. (3.17),

$$(\mathbf{A}\times\hat{\mathbf{y}})\cdot\hat{\mathbf{z}} = (\hat{\mathbf{x}}3+\hat{\mathbf{z}})\cdot\hat{\mathbf{z}} = 1.$$

Eq. (3.29) and Eq. (3.25) could also have been used in the solution.

Problem 3.22 Use the appropriate expression for the differential surface area ds to determine the area of each of the following surfaces:

(a) $r=3;\ 0\le\phi\le\pi/3;\ -2\le z\le 2,$

(b) $2\le r\le 5;\ \pi/2\le\phi\le\pi;\ z=0,$

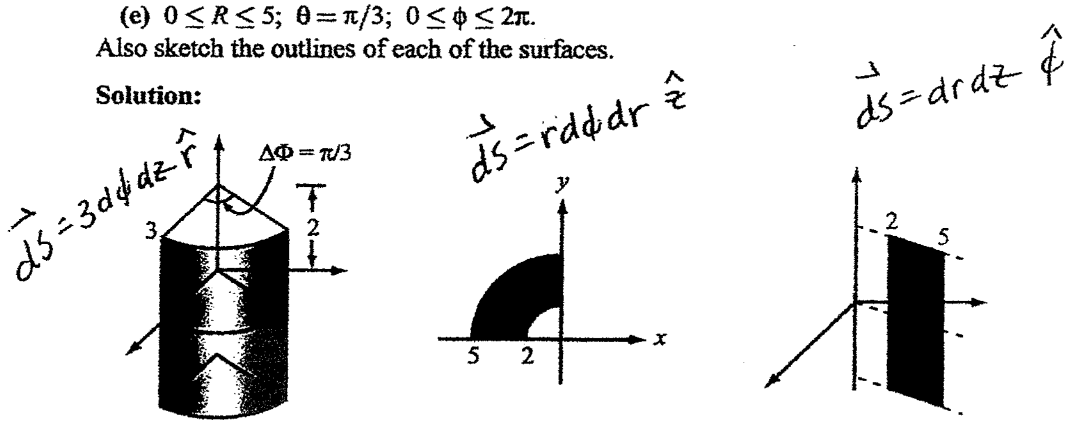
(c) $2\le r\le 5;\ \phi=\pi/4;\ -2\le z\le 2,$

(d) $R=2;\ 0\le\theta\le\pi/3;\ 0\le\phi\le\pi,$

(e) $0\le R\le 5;\ \theta=\pi/3;\ 0\le\phi\le 2\pi.$

Also sketch the outlines of each of the surfaces.

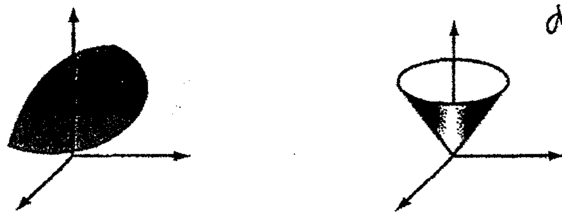
Solution:



(a) (b) (c)

$\vec{ds} = 2^2 \sin\theta \, d\theta \, d\phi \, \hat{R}$

$\vec{ds} = R \sin(\pi/3) \, d\phi \, dR \, \hat{\theta}$



(d) (e)

Figure P3.22: Surfaces described by Problem 3.22.

(a) Using Eq. (3.43a),

$$A = \int_{z=-2}^{2} \int_{\phi=0}^{\pi/3} (r)\big|_{r=3} \, d\phi \, dz = \left((3\phi z)\big|_{\phi=0}^{\pi/3} \right)\Big|_{z=-2}^{2} = 4\pi.$$

(b) Using Eq. (3.43c),

$$A = \int_{r=2}^{5} \int_{\phi=\pi/2}^{\pi} (r)\big|_{z=0} \, d\phi \, dr = \left(\left(\tfrac{1}{2} r^2 \phi\right)\Big|_{r=2}^{5} \right)\Big|_{\phi=\pi/2}^{\pi} = \frac{21\pi}{4}.$$

(c) Using Eq. (3.43b),

$$A = \int_{z=-2}^{2} \int_{r=2}^{5} (1)\big|_{\phi=\pi/4} \, dr \, dz = \left((rz)\big|_{z=-2}^{2} \right)\Big|_{r=2}^{5} = 12.$$

(d) Using Eq. (3.50b),

$$A = \int_{\theta=0}^{\pi/3} \int_{\phi=0}^{\pi} \left(R^2 \sin\theta\right)\big|_{R=2} \, d\phi \, d\theta = \left((-4\phi \cos\theta)\big|_{\theta=0}^{\pi/3} \right)\Big|_{\phi=0}^{\pi} = 2\pi.$$

(e) Using Eq. (3.50c),

$$A = \int_{R=0}^{5} \int_{\phi=0}^{2\pi} (R \sin\theta)\big|_{\theta=\pi/3} \, d\phi \, dR = \left(\left(\tfrac{1}{2} R^2 \phi \sin\frac{\pi}{3} \right)\Big|_{\phi=0}^{2\pi} \right)\Bigg|_{R=0}^{5} = \frac{25\sqrt{3}\pi}{2}.$$

Problem 3.23 Find the volumes described by
(a) $2 \le r \le 5$; $\pi/2 \le \phi \le \pi$; $0 \le z \le 2$,
(b) $0 \le R \le 5$; $0 \le \theta \le \pi/3$; $0 \le \phi \le 2\pi$.
Also sketch the outline of each volume.

Solution:

$dV = r\,d\phi\,dr\,dz$

$dV = R^2 \sin\theta\, d\phi\, d\theta\, dR$

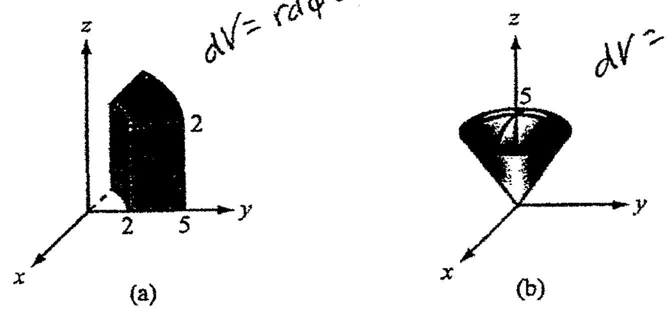


Figure P3.23: Volumes described by Problem 3.23 .

(a) From Eq. (3.44),

$$V = \int_{z=0}^{2} \int_{\phi=\pi/2}^{\pi} \int_{r=2}^{5} r\,dr\,d\phi\,dz = \left(\left(\left(\tfrac{1}{2} r^2 \phi z\right)\Big|_{r=2}^{5} \right)\Big|_{\phi=\pi/2}^{\pi} \right)\Big|_{z=0}^{2} = \frac{21\pi}{2}.$$

(b) From Eq. (3.50e),

$$V = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/3} \int_{R=0}^{5} R^2 \sin\theta\, dR\, d\theta\, d\phi$$

$$= \left(\left(\left(-\cos\theta \frac{R^3}{3} \phi \right)\Big|_{R=0}^{5} \right)\Big|_{\theta=0}^{\pi/3} \right)\Big|_{\phi=0}^{2\pi} = \frac{125\pi}{3}.$$

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Problem 3.26 At a given point in space, vectors **A** and **B** are given in spherical coordinates by

$$\mathbf{A} = \hat{\mathbf{R}}4 + \hat{\boldsymbol{\theta}}2 - \hat{\boldsymbol{\phi}},$$
$$\mathbf{B} = -\hat{\mathbf{R}}2 + \hat{\boldsymbol{\phi}}3.$$

Find:

(a) the scalar component, or projection, of **B** in the direction of **A**,

(b) the vector component of **B** in the direction of **A**,

(c) the vector component of **B** perpendicular to **A**.

Solution:

(a) Scalar component of **B** in direction of **A**:

$$C = \mathbf{B}\cdot\hat{\mathbf{a}} = \mathbf{B}\cdot\frac{\mathbf{A}}{|\mathbf{A}|} = (-\hat{\mathbf{R}}2 + \hat{\boldsymbol{\phi}}3)\cdot\frac{(\hat{\mathbf{R}}4 + \hat{\boldsymbol{\theta}}2 - \hat{\boldsymbol{\phi}})}{\sqrt{16+4+1}}$$
$$= \frac{-8-3}{\sqrt{21}} = -\frac{11}{\sqrt{21}} = -2.4.$$

(b) Vector component of **B** in direction of **A**:

$$\mathbf{C} = \hat{\mathbf{a}}C = \mathbf{A}\frac{C}{|\mathbf{A}|} = (\hat{\mathbf{R}}4 + \hat{\boldsymbol{\theta}}2 - \hat{\boldsymbol{\phi}})\frac{(-2.4)}{\sqrt{21}}$$
$$= -(\hat{\mathbf{R}}2.09 + \hat{\boldsymbol{\theta}}1.05 - \hat{\boldsymbol{\phi}}0.52).$$

(c) Vector component of **B** perpendicular to **A**:

$$\mathbf{D} = \mathbf{B} - \mathbf{C} = (-\hat{\mathbf{R}}2 + \hat{\boldsymbol{\phi}}3) + (\hat{\mathbf{R}}2.09 + \hat{\boldsymbol{\theta}}1.05 - \hat{\boldsymbol{\phi}}0.52)$$
$$= \hat{\mathbf{R}}0.09 + \hat{\boldsymbol{\theta}}1.05 + \hat{\boldsymbol{\phi}}2.48.$$